

BLOCKS OF SMALL DEFECT

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ABSTRACT. Let G be a finite solvable group, let p be a prime such that $p \geq 5$ and $O_p(G) = 1$, and we denote $|G|_p = p^n$, then G contains a block of defect less than or equal to $\lfloor \frac{3n}{5} \rfloor$. Let G be a finite solvable group and let p^a be the largest power of p dividing $\chi(1)$ for an irreducible character χ of G , we show that $|G : \mathbf{F}(G)|_p \leq p^{3a}$ for $p \geq 5$.

1. INTRODUCTION

It is a classical theme to study orbit structures of a finite group G over a finite, faithful and completely reducible G -module V . One of the most important and natural questions about orbit structure is to establish the existence of an orbit of a certain size. For a long time, there has been a deep interest and need to examine the size of the largest possible orbits in linear group actions. In 2004, A. Moretó and T.R. Wolf [22] studied the solvable linear groups and investigated the relation of the point stabilizers $\mathbf{C}_G(v)$ and the Fitting subgroup series of G . They proved an important orbit theorem [22, Theorem E] and applied it to obtain results showing that solvable groups have large character degrees and conjugacy classes. This orbit theorem and its consequences were used to obtain a number of results on several conjectures on class sizes, character degrees and zeros of characters.

In [27], Yang strengthened this result by showing the following. Suppose that V is a faithful completely reducible G -module where G is a finite solvable group, then there exists $v \in V$ and $K \triangleleft G$ such that $\mathbf{C}_G(v) \subseteq K$, where the Fitting length of K is less than or equal to 7. An example [27, Section 4] was provided to show that the improvement is the best possible. Although one cannot say more in general because of this example, it is possible to show that there exists an element $v \in V$ such that the p -part of $\mathbf{C}_G(v)$ is relatively small for all the primes $p \geq 5$. In this paper, we show the following orbit theorem of solvable linear groups. This theorem in some sense is the best possible as the semi-linear group shows us.

Theorem A. *Let π_0 be the set of all the primes except 2 and 3. Let G be a finite solvable group and let V be a finite, faithful and completely reducible G -module (possibly of mixed characteristic). Then there exists $K \triangleleft G$, $K \subseteq \mathbf{F}_2(G)$ and there exist two G -orbits with representatives $v_a, v_b \in V$ such that for any $H \in \text{Hall}_{\pi_0}(G)$, we have $\mathbf{C}_H(v_a) \subseteq K$ and $\mathbf{C}_H(v_b) \subseteq K$. The Hall π_0 -subgroup of $K\mathbf{F}(G)/\mathbf{F}(G)$ and the Hall π_0 -subgroup of $K \cap \mathbf{F}(G)$ are abelian.*

Theorem A can be used to study the following problems although they look different at the first glance.

Let G be a finite group. Let p be a prime and $|G|_p = p^n$. An irreducible ordinary character of G is called p -defect 0 if and only if its degree is divisible by p^n . It is an interesting problem to give necessary and sufficient conditions for the existence of p -blocks of defect zero. If a finite group G has a character of p -defect 0, then $O_p(G) = 1$ [6, Corollary 6.9]. Unfortunately,

the converse is not true. Although the block of defect zero may not exist in general, one could try to find the smallest defect $d(B)$ of a block B of G . One of the results along this line is proved by Espuelas and Navarro [5, Theorem A]. Let G be a (solvable) group of odd order such that $O_p(G) = 1$ and $|G|_p = p^n$, then G contains a p -block B such that $d(B) \leq \lfloor n/2 \rfloor$. The bound is best possible, as shown by an example in [5]. In the same paper, they raised the following question. If G is a finite group with $O_p(G) = 1$, $p \geq 5$, and $|G|_p = p^n$, does G contain a block of defect less than $\lfloor \frac{n}{2} \rfloor$?

Using Theorem A, we prove the following result as a partial answer to this question. The bound we obtain here is pretty sharp since $\lfloor \frac{n}{2} \rfloor$ is the best one may get.

Theorem B. *Let G be a finite solvable group, let p be a prime such that $p \geq 5$ and $O_p(G) = 1$, and we denote $|G|_p = p^n$. Then G contains a p -block B such that $d(B) \leq \lfloor \frac{3n}{5} \rfloor$.*

Let p^a denote the largest power of p dividing $\chi(1)$ for an irreducible character χ of G . Moretó and Wolf [22, Theorem A] proved that for G solvable, there exists a product $\theta = \chi_1(1) \cdots \chi_t(1)$ of distinct irreducible characters χ_i such that $|G : \mathbf{F}(G)|$ divides $\theta(1)$ and $t \leq 19$. This implies that $|G : \mathbf{F}(G)|_p \leq p^{19a}$. They also suggest that a better bound $|G : \mathbf{F}(G)|_p \leq p^{2a}$ might be true for all solvable groups. In fact, they believe [22, Question 2.2] that for solvable groups one may find two irreducible characters χ_1 and χ_2 such that $|G : \mathbf{F}(G)| \mid \chi_1(1)\chi_2(1)$. Although it is difficult to answer this question in general, we are able to prove a closely related result using the previous orbit theorem. As a corollary, we show that $|G : \mathbf{F}(G)|_p \leq p^{3a}$ for $p \geq 5$.

The Huppert's $\rho - \sigma$ conjectures state that there is an irreducible character χ of G and a conjugacy class C of G such that the degree of χ and $|C|$ are each divisible by many primes. For the character theoretic $\rho - \sigma$ problem, we define $\rho(G)$ be those primes that divide the degree of some irreducible character of G and $\sigma(G)$ be the maximum number of primes dividing the degree of an irreducible character of G . Huppert conjectures that $|\rho(G)|$ can be bounded in terms of $\sigma(G)$, and if G is solvable, then even $|\rho(G)| \leq 2\sigma(G)$. Up to now the best known bound for G solvable is $|\rho(G)| \leq 3\sigma(G) + 2$ by Manz of Wolf [18, Theorems 1.4]. Theorem A may be used to study Huppert's $\rho - \sigma$ conjectures and obtain the best known bound.

Theorem A also has connections to other questions about degrees of characters and lengths of conjugacy classes of solvable groups.

2. NOTATION AND LEMMAS

Notation:

- (1) Let G be a finite group, let S be a subset of G and let π be a set of different primes. For each prime p , we denote $\text{SP}_p(S) = \{\langle x \rangle \mid o(x) = p, x \in S\}$ and $\text{EP}_p(S) = \{x \mid o(x) = p, x \in S\}$. We denote $\text{SP}(S) = \bigcup_p \text{primes } \text{SP}_p(S)$, $\text{SP}_\pi(S) = \bigcup_{p \in \pi} \text{SP}_p(S)$, $\text{EP}(S) = \bigcup_p \text{primes } \text{EP}_p(S)$ and $\text{EP}_\pi(S) = \bigcup_{p \in \pi} \text{EP}_p(S)$. We denote $\text{NEP}(S) = |\text{EP}(S)|$, $\text{NEP}_p(S) = |\text{EP}_p(S)|$ and $\text{NEP}_\pi(S) = |\text{EP}_\pi(S)|$. We denote $\text{NSP}(S) = |\text{SP}(S)|$, $\text{NSP}_p(S) = |\text{SP}_p(S)|$ and $\text{NSP}_\pi(S) = |\text{SP}_\pi(S)|$.
- (2) Let n be an even integer, q a power of a prime. Let V be a standard symplectic vector space of dimension n of \mathbb{F}_q . We use $\text{SCRSp}(n, q)$ or $\text{SCRSp}(V)$ to denote the set of all solvable subgroups of $\text{Sp}(V)$ which acts completely reducibly on V . We use $\text{SIRSp}(n, q)$ or $\text{SIRSp}(V)$ to denote the set of all solvable subgroups of $\text{Sp}(V)$

which acts irreducibly on V . Define $\text{SCRSp}(n_1, q_1) \times \text{SCRSp}(n_2, q_2) = \{H \times I \mid H \in \text{SCRSp}(n_1, q_1) \text{ and } I \in \text{SCRSp}(n_2, q_2)\}$.

- (3) If V is a finite vector space of dimension n over $\text{GF}(q)$, where q is a prime power, we denote by $\Gamma(q^n) = \Gamma(V)$ the semi-linear group of V , i.e.,

$$\Gamma(q^n) = \{x \mapsto ax^\sigma \mid x \in \text{GF}(q^n), a \in \text{GF}(q^n)^\times, \sigma \in \text{Gal}(\text{GF}(q^n)/\text{GF}(q))\},$$

and we define

$$\Gamma_0(q^n) = \{x \mapsto ax \mid x \in \text{GF}(q^n), a \in \text{GF}(q^n)^\times\}.$$

- (4) We use $\mathbf{F}(G)$ to denote the Fitting subgroup of G . Let $\mathbf{F}_0(G) \leq \mathbf{F}_1(G) \leq \mathbf{F}_2(G) \leq \dots \leq \mathbf{F}_n(G) = G$ denote the ascending Fitting series, i.e. $\mathbf{F}_0(G) = 1$, $\mathbf{F}_1(G) = \mathbf{F}(G)$ and $\mathbf{F}_{i+1}(G)/\mathbf{F}_i(G) = \mathbf{F}(G/\mathbf{F}_i(G))$. $\mathbf{F}_i(G)$ is the i th ascending Fitting subgroup of G .
- (5) Let π_0 be the set of all the primes except 2 and 3.

Definition 2.1. Suppose that a finite solvable group G acts faithfully, irreducibly and quasi-primitively on a finite vector space V . Let $\mathbf{F}(G)$ be the Fitting subgroup of G and $\mathbf{F}(G) = \prod_i P_i$, $i = 1, \dots, m$ where P_i are normal p_i -subgroups of G for different primes p_i . Let $Z_i = \Omega_1(\mathbf{Z}(P_i))$. We define

$$E_i = \begin{cases} \Omega_1(P_i) & \text{if } p_i \text{ is odd;} \\ [P_i, G, \dots, G] & \text{if } p_i = 2 \text{ and } [P_i, G, \dots, G] \neq 1; \\ Z_i & \text{otherwise.} \end{cases}$$

By possible reordering we may assume that $E_i \neq Z_i$ for $i = 1, \dots, s$, $0 \leq s \leq m$ and $E_i = Z_i$ for $i = s+1, \dots, m$. We define $E = \prod_{i=1}^s E_i$, $Z = \prod_{i=1}^s Z_i$ and we define $\bar{E}_i = E_i/Z_i$, $\bar{E} = E/Z$. Furthermore, we define $e_i = \sqrt{|E_i/Z_i|}$ for $i = 1, \dots, s$ and $e = \sqrt{|E/Z|}$.

Theorem 2.2, Lemma 2.3, Lemma 2.7 and Lemma 2.8 are proved in [28] and [30] but we include the proofs here for completeness.

Theorem 2.2. Suppose that a finite solvable group G acts faithfully, irreducibly and quasi-primitively on an n -dimensional finite vector space V over finite field \mathbb{F} of characteristic r . We use the notation in Definition 2.1. Then every normal abelian subgroup of G is cyclic and G has normal subgroups $Z \leq U \leq F \leq A \leq G$ such that,

- (1) $F = EU$ is a central product where $Z = E \cap U = \mathbf{Z}(E)$ and $\mathbf{C}_G(F) \leq F$;
- (2) $F/U \cong E/Z$ is a direct sum of completely reducible G/F -modules;
- (3) E_i is an extra-special p_i -group for $i = 1, \dots, s$ and $e_i = p_i^{n_i}$ for some $n_i \geq 1$. Furthermore, $(e_i, e_j) = 1$ when $i \neq j$ and $e = e_1 \dots e_s$ divides n , also $\gcd(r, e) = 1$;
- (4) $A = \mathbf{C}_G(U)$ and $G/A \lesssim \text{Aut}(U)$, A/F acts faithfully on E/Z ;
- (5) $A/\mathbf{C}_A(E_i/Z_i) \lesssim \text{Sp}(2n_i, p_i)$;
- (6) U is cyclic and acts fixed point freely on W where W is an irreducible submodule of V_U ;
- (7) $|V| = |W|^{eb}$ for some integer b ;
- (8) $|G : A| \mid \dim(W)$. Assume $g \in G \setminus A$ and $o(g) = s$ where s is a prime, then $|\mathbf{C}_V(g)| = |W|^{\frac{1}{s}eb}$;
- (9) G/A is cyclic.

Proof. By [19, Theorem 1.9] there exist $\tilde{E}_i, T_i \triangleleft G$ and all the following hold,

- i) $P_i = \tilde{E}_i T_i$, $\tilde{E}_i \cap T_i = Z_i$ and $T_i = \mathbf{C}_{P_i}(\tilde{E}_i)$;
- ii) \tilde{E}_i is extra-special or $\tilde{E}_i = Z_i$;
- iii) $\exp(\tilde{E}_i) = p_i$ or $p_i = 2$;
- iv) T_i is cyclic, or $p_i = 2$ and T_i is dihedral, quaternion or semidihedral;
- v) If T_i is not cyclic, then there exists $U_i \triangleleft G$ with U_i cyclic, $U_i \leq T_i$, $|T_i : U_i| = 2$ and $\mathbf{C}_{T_i}(U_i) = U_i$;
- vi) If $\tilde{E}_i > Z_i$, then $E_i/Z_i = E_{i1}/Z_i \times \cdots \times E_{id}/Z_i$ for chief factors E_{ik}/Z_i of G and with $Z_i = \mathbf{Z}(E_{ik})$ for each k and $E_{ik} \leq \mathbf{C}_G(E_{il})$ for $k \neq l$.

We define $U_i = T_i$ if $p_i \neq 2$. We define $U = \prod_{i=1}^m U_i$, $T = \prod_{i=1}^m T_i$, $F = EU$ and $A = \mathbf{C}_G(U)$.

If $p_i \neq 2$, then by (i),(ii),(iii) $\tilde{E}_i = \Omega_1(P_i)$ and therefore $\tilde{E}_i = E_i$. If $p_i = 2$ and assume $\tilde{E}_i > Z_i$, $\tilde{E}_i/Z_i = \prod_k E_{ik}/Z_i$ for chief factors E_{ik}/Z_i of G by (vi) and thus $E_{ik} = [E_{ik}, G]$ and $\tilde{E}_i = [\tilde{E}_i, G]$. By (v), $[T_i, G, \dots, G] = 1$. Thus $[P_i, G, G, \dots, G] = \tilde{E}_i$ and therefore $\tilde{E}_i = E_i$.

The other results mainly follow from Corollary 1.10, 2.6 and Lemma 2.10 of [19]. Since $\mathbf{C}_G(F) = \mathbf{C}_G(EU) \leq \mathbf{C}_G(E) = T$ and $\mathbf{C}_T(U) = U$, we have $\mathbf{C}_G(F) \leq F$. Since $A = \mathbf{C}_G(U)$, $\mathbf{F}(G) \cap A = \mathbf{C}_{\mathbf{F}(G)}(U) = EU = F$ and thus A/F acts faithfully on E/Z .

Let \mathbb{K} be the algebraic closure of \mathbb{F} , then $W \otimes_{\mathbb{F}} \mathbb{K} = W_1 \oplus W_2 \oplus \cdots \oplus W_m$, where the W_i are Galois conjugate, non-isomorphic irreducible U -modules. In particular, each W_i is faithful, $\dim_{\mathbb{K}} W_i = 1$. Clearly $\mathbf{N}_G(W_i) \geq \mathbf{C}_G(U)$ for each i . Furthermore, $[\mathbf{N}_G(W_i), U] \leq \mathbf{C}_U(W_i)$ since U is normal. Thus $\mathbf{N}_G(W_i) = \mathbf{C}_G(U) = A$. It follows that G/A permutes the set $\{W_1, \dots, W_m\}$ in orbits of length $|G : A|$ and thus $|G : A| \mid \dim(W)$. Since G/A permutes the W_i fixed point freely, for all $g \in G \setminus A$ of order s where s is a prime, $|\mathbf{C}_V(g)| = |W|^{\frac{1}{s}eb}$. This proves (8).

The fact that G/A is cyclic is essentially proved in [24, Section 20]. We give an argument here for completeness. $\mathbb{F}U$ is a semi-simple algebra over \mathbb{F} and $\dim(\mathbb{F}U) = |U|$. By Wedderburn's Theorem, there exist a finite number of idempotents e_1, \dots, e_α and $e_i \mathbb{F}U$ is isomorphic to a full matrix algebra over some division algebra D over \mathbb{F} . Since $\mathbb{F}U$ is commutative, the division algebra D is actually a field and the dimension of the matrix is 1. It follows that $e_i \mathbb{F}U$ is a field for all $i = 1, \dots, \alpha$. Set $K_i = e_i \mathbb{F}U$, then K_1, \dots, K_α are fields and $\mathbb{F}U = K_1 \oplus \cdots \oplus K_\alpha$.

Since V is a quasi-primitive G -module, V_U is a homogeneous $\mathbb{F}U$ -module. Then there exists some $j \in \{1, \dots, \alpha\}$ such that $e_j V = V$ and $e_l V = 0$ for any $l \neq j$. Thus V is a K_j vector space. K_j is a finite field extension of $\mathbb{F}e_j$. Let $g \in G$, then $gK_jg^{-1} = K_j$ because V is quasi-primitive. Define $\sigma_g : K_j \rightarrow K_j$ as $\sigma_g(\beta) = g\beta g^{-1}$. σ_g is a field automorphism of K_j and fixes every element of $\mathbb{F}e_j$. Thus $\sigma_g \in \text{Gal}(K_j/\mathbb{F}e_j)$. We claim that there is a natural map φ between $G \mapsto \text{Gal}(K_j/\mathbb{F}e_j)$ defined by $\varphi(g) = \sigma_g$. Clearly φ is a group homomorphism and $\mathbf{C}_G(U) \subseteq \ker(\varphi)$. Suppose that $g \in G$, $g \notin \mathbf{C}_G(U)$ and let $u_0 \in U$ be a generator for U . Then $u_0 \neq gu_0g^{-1}$ and the action of u_0 on V is different than the action of gu_0g^{-1} on V since the action of G on V is faithful. Thus the action of $e_j u_0$ on V is different than the action of $e_j gu_0g^{-1}$ on V and we know that $\sigma_g(e_j u_0) \neq e_j u_0$. Thus we have $\ker(\varphi) = \mathbf{C}_G(U) = A$. This proves (9). \square

Lemma 2.3. *Suppose that a finite solvable group G acts faithfully, irreducibly and quasi-primitively on a finite vector space V . Using the notation in Theorem 2.2, we have $|G| \mid \dim(W) \cdot |A/F| \cdot e^2 \cdot (|W| - 1)$.*

Proof. By Theorem 2.2, $|G| = |G/A||A/F||F|$ and $|F| = |E/Z||U|$. Since $|G/A| \mid \dim(W)$, $|E/Z| = e^2$ and $|U| \mid (|W| - 1)$, we have $|G| \leq \dim(W) \cdot |A/F| \cdot e^2 \cdot (|W| - 1)$. \square

Lemma 2.4. *Suppose that a finite solvable group G acts faithfully and quasi-primitively on a finite vector space V over the field \mathbb{F} . Let $g \in \text{EP}_s(G)$ where s is a prime and we use the notation in Theorem 2.2.*

- (1) *If $g \in F$ then $|\mathbf{C}_V(g)| \leq |W|^{\frac{1}{2}eb}$.*
- (2) *If $g \in A \setminus F$, $s \geq 5$ and $s \nmid |E|$, then $|\mathbf{C}_V(g)| \leq |W|^{\lfloor \frac{1}{3}e \rfloor b}$.*
- (3) *If $g \in G \setminus A$ then $|\mathbf{C}_V(g)| \leq |W|^{\frac{1}{s}eb}$.*

Proof. It is proved in [19, Proposition 4.10] that for $g \in \mathbf{F}(G)$, $|\mathbf{C}_V(g)| \leq |W|^{\frac{1}{2}eb} = |V|^{1/2}$. Since $F \leq \mathbf{F}(G)$, (1) follows.

Since $\mathbf{C}_G(F) \leq F$ and $g \notin F$, $[g, F] \neq 1$. Since $g \in A = \mathbf{C}_G(U)$ and $F = EU$, $[g, E] \neq 1$. Since $s \nmid |E|$, there exists a g -invariant q -subgroup $Q \leq E$, for some prime $q \neq s$ such that Q is extra-special, $[Q, g] = Q$, $[\mathbf{Z}(Q), g] = 1$, $\mathbf{Z}(Q) \triangleleft G$ and the action of g on $Q/\mathbf{Z}(Q)$ is fixed-point free. Let \mathbb{K} be a splitting field for $\langle g \rangle Q$ which is a finite extension of \mathbb{F} and set $V_{\mathbb{K}} = V \otimes_{\mathbb{F}} \mathbb{K}$. Since $\dim_{\mathbb{K}}(\mathbf{C}_{V_{\mathbb{K}}}(g)) = \dim_{\mathbb{F}}(\mathbf{C}_V(g))$, we may consider $V_{\mathbb{K}}$ instead of V . Let $0 = V_0 \subset V_1 \subset \dots \subset V_l = V_{\mathbb{K}}$ be a $\langle g \rangle Q$ -composition series for $V_{\mathbb{K}}$ with quotient $\overline{V}_j = V_j/V_{j-1}$ for $j = 1, \dots, l$. Thus each \overline{V}_j is an absolute irreducible $\langle g \rangle Q$ module. Since $V_{\mathbb{K}}$ is obtained by tensoring a quasi-primitive module up to a splitting field, $V_{\mathbb{K}}|_{\mathbf{Z}(Q)}$ is a direct sum of Galois conjugate irreducible modules, $\mathbf{Z}(Q)$ is faithful on every irreducible summand of $V_{\mathbb{K}}|_{\mathbf{Z}(Q)}$. By the Jordan-Holder Theorem, these are the only irreducibles that can occur in $\overline{V}_j|_{\mathbf{Z}(Q)}$ and thus $\mathbf{Z}(Q)$ acts faithfully on \overline{V}_j . Since all nontrivial normal subgroups of $\langle g \rangle Q$ contain $\mathbf{Z}(Q)$, $\langle g \rangle Q$ is faithful on \overline{V}_j . Since g centralize $\mathbf{Z}(Q)$, \overline{V}_j is also an irreducible Q module. Let $|Q| = q^{2k+1}$, then by Schult's Theorem [13, Theorem V.17.13] or Hall-Higman Theorem [14, Theorem IX.2.6], $\dim_{\mathbb{K}}(\overline{V}_j) = q^k \mid e$ and $\dim_{\mathbb{K}}(\mathbf{C}_{\overline{V}_j}(g)) \leq \lfloor \beta \cdot q^k \rfloor$ where,

$$\beta = \begin{cases} \frac{1}{s} \left(\frac{q^{k+s}-1}{q^k-1} \right) & \text{if } s \mid q^k - 1; \\ \frac{1}{s} \left(\frac{q^{k+1}-1}{q^k-1} \right) & \text{if } s \mid q^k + 1. \end{cases}$$

Assume $s \mid q^k - 1$ and let $q^k - 1 = st$ where $t \geq 1$ is an integer, then $\beta = \frac{t+1}{st+1}$. Thus $\beta \leq \frac{1}{3}$ when $s \geq 5$.

Assume $s \mid q^k + 1$ and let $q^k + 1 = st$ where $t \geq 1$ is an integer, then $\beta = \frac{t}{st-1}$. Thus $\beta \leq \frac{1}{3}$ when $s \geq 5$.

Since $\dim_{\mathbb{K}}(\mathbf{C}_{V_{\mathbb{K}}}(g)) \leq \sum_j \dim_{\mathbb{K}}(\mathbf{C}_{\overline{V}_j}(g))$, (2) holds.

(3) follows from Theorem 2.2(8). \square

Lemma 2.5. *Assume G satisfies Theorem 2.2 and we adopt the notation in it. Let p be a prime and $x \in \text{EP}_p(A \setminus F)$ and assume $|\mathbf{C}_{E/Z}(x)| = \prod_i p_i^{m_i}$. We have the following:*

- (1) $\text{NEP}_p(A \setminus F) \leq \text{NEP}_p(A/F)|F|$.
- (2) $\text{NEP}_p(A \setminus F) \leq \frac{\text{NEP}_p(A/F)|F|}{\prod_{p_i \neq p} p_i^{m_i}}$.

Proof. This follows from [28, Lemma 2.7]. \square

Lemma 2.6. *Assume that A is a normal subgroup of G and G/A is cyclic. Then we have $\text{NSP}_{\pi_0}(G \setminus A) \leq \lfloor \log_5(|G/A|) \rfloor \cdot |A|$.*

Proof. Since G/A is cyclic,

$$\text{NSP}_{\pi_0}(G \setminus A) \leq \sum_{p \in \pi_0, p \mid |G/A|} (p-1) \cdot |A|/(p-1) = \sum_{p \in \pi_0, p \mid |G/A|} |A| \leq \log_5(|G/A|) \cdot |A|.$$

□

Lemma 2.7. *Let V be a symplectic vector space of dimension n with base field \mathbb{F} and $G \in \text{SIRSp}(n, \mathbb{F})$. Assume further the action is not quasi-primitive and that $N \triangleleft G$ is maximal such that V_N is not homogeneous. Let $V_N = V_1 \oplus \cdots \oplus V_t$ where V_i 's are the homogeneous N -modules and clearly $t \geq 2$. Then either all V_i are non-singular or all are totally isotropic. In the first case, $\dim(V_i)$ is even, $G \lesssim H \wr S$ as linear groups where $H \in \text{SIRSp}(V_1)$. In the second case $t = 2$, V_2 is isomorphic to V_1^* as an N -module, and we say that V_N is a pair.*

Proof. V is a symplectic G -module with respect to the non-singular symplectic form $(\ , \)$. By [19, Proposition 0.2], $S = G/N$ faithfully and primitively permutes the homogeneous components of V_N . Set $I = \mathbf{N}_G(V_1)$, by Clifford's Theorem, V_1 is an irreducible I -module. Since the form $(\ , \)$ is G -invariant, the subspace $\{v \in V_1 \mid (v, v') = 0 \text{ for all } v' \in V_1\}$ is an I -submodule of V_1 and the form $(\ , \)$ is either totally isotropic or non-singular on V_1 . Since G transitively permutes the V_i , the G -invariant form $(\ , \)$ is simultaneously totally isotropic or non-singular on all the V_i . If the form is non-singular on each V_j , then let $H = \mathbf{N}_G(V_1)/\mathbf{C}_G(V_1)$ and we know $H \in \text{SIRSp}(V_1)$, $G \lesssim H \wr S$ as linear groups. Hence, we assume that each V_i is totally isotropic.

Let $j \in \{1, \dots, t\}$. Then, set $V_j^\perp = \{v \in V \mid (v, v_j) = 0 \text{ for all } v_j \in V_j\}$. For $v \in V$, we consider the map $f_v \in V_j^* := \text{Hom}_{\mathbb{F}}(V_j, \mathbb{F})$, defined by $f_v(v_j) = (v, v_j)$, $v_j \in V_j$. Then $v \mapsto f_v$, $v \in V$, induces a N -isomorphism between V/V_j^\perp and the dual space V_j^* . Since V_N is completely reducible, there exists an N -module U_j such that $V_N = V_j^\perp \oplus U_j$. Thus $U_j \cong V_j^*$ is homogeneous and $U_j = V_{\pi(j)}$ for a permutation $\pi \in S_t$. Then π is an involution in S_t and the permutation action of S commutes with π . Hence, S acts on the orbits of π . Since π is not the identity, and the action of S is primitive, it follows that π has a single orbit and $t = 2$. □

Lemma 2.8. *Let V be a symplectic vector space of dimension $2n$ with base field \mathbb{F} and $G \in \text{SIRSp}(2n, \mathbb{F})$, $|\mathbb{F}| = p$ where p is a prime. Assume G acts irreducibly and quasi-primitively on V and $e = 1$, then we have the following:*

- (1) $G \lesssim \Gamma(p^{2n})$. G/U is cyclic and $|G/U| \mid 2n$.
- (2) $U \lesssim \Gamma_0(p^{2n})$ and $|U| \mid p^n + 1$.

Proof. By [25, Proposition 3.1(1)], G may be identified with a subgroup of the semi-direct product of $\text{GF}(p^{2n})^\times$ by $\text{Gal}(\text{GF}(p^{2n}) : \text{GF}(p))$ acting in a natural manner on $\text{GF}(p^{2n})^+$. Also $G \cap \text{GF}(p^{2n})^\times = U$ and $|G \cap \text{GF}(p^{2n})^\times| \mid p^n + 1$. Clearly G/U is cyclic of order dividing $2n$. Now (1) and (2) hold. □

Lemma 2.9. *Let V be a finite, faithful irreducible G -module and G is solvable. Suppose V is a vector space of dimension n over the field \mathbb{F} . Let $(n, \mathbb{F}) = (4, \mathbb{F}_2)$, then $|G| \leq 6^2 \cdot 2$ and $\text{NEP}_{\{2,3\}}(G) \leq 4$. Also G_{π_0} is abelian and $G_{\pi_0} \subseteq \mathbf{F}(G)$.*

Proof. G satisfies one of the following:

- (1) $G \lesssim S_3 \wr S_2$. Clearly $|G| \mid 6^2 \cdot 2$ and $G_{\pi_0} = 1$.

- (2) V is irreducible and quasi-primitive, $e = 1$ and $G \lesssim \Gamma(2^4)$. $|G| \mid 60$, $G_{\pi_0} \lesssim Z_5$ and $G_{\pi_0} \subseteq \mathbf{F}(G)$. Clearly $\text{NEP}_{\{2,3\}'}(G) \leq 4$.

Hence the result holds in all cases. \square

Lemma 2.10. *Let n be an even integer and V be a symplectic vector space of dimension n of field \mathbb{F} . Let $G \in \text{SCRSp}(n, \mathbb{F})$.*

- (1) *Let $(n, \mathbb{F}) = (2, \mathbb{F}_2)$, then $G \lesssim S_3$ and $|G| \mid 6$.*
- (2) *Let $(n, \mathbb{F}) = (4, \mathbb{F}_2)$, then $|G| \leq 6^2 \cdot 2$ and $\text{NEP}_{\pi_0}(G) \leq 4$. Also G_{π_0} is abelian and $G_{\pi_0} \subseteq \mathbf{F}(G)$.*
- (3) *Let $(n, \mathbb{F}) = (6, \mathbb{F}_2)$, then $|G| \leq 6^4$, $\text{NEP}_{\pi_0}(G) \leq 6$. Also G_{π_0} is abelian and $G_{\pi_0} \subseteq \mathbf{F}(G)$.*
- (4) *Let $(n, \mathbb{F}) = (8, \mathbb{F}_2)$, then $|G| \leq 6^4 \cdot 24$ and $\text{NEP}_{\pi_0}(G) \leq 24$.*
- (5) *Let $(n, \mathbb{F}) = (2, \mathbb{F}_3)$, then $G \lesssim \text{SL}(2, 3)$ and $|G| \mid 24$.*
- (6) *Let $(n, \mathbb{F}) = (4, \mathbb{F}_3)$, then $|G| \leq 24^2 \cdot 2$, $\text{NEP}_5(G) \leq 64$ and G has no elements with prime order $p \geq 7$. If $5 \mid |G|$, then $|G| \leq 320$.*

Proof. V is a symplectic G -module with respect to the non-singular symplectic form $(\ , \)$. If V is not irreducible, we may choose an irreducible submodule W of V of smallest possible dimension and set $\dim(W) = m$. Since the form $(\ , \)$ is G -invariant, the subspace $\{v \in W \mid (v, v') = 0 \text{ for all } v' \in W\}$ is a submodule of W and the form $(\ , \)$ is either totally isotropic or non-singular on W . If the form $(\ , \)$ is totally isotropic on W , then set $W^\perp = \{v \in V \mid (v, w) = 0 \text{ for all } w \in W\}$. For $v \in V$, we consider the map $f_v \in W^* := \text{Hom}_{\mathbb{F}}(W, \mathbb{F})$, defined by $f_v(w) = (v, w)$, $w \in W$. Then $v \mapsto f_v$, $v \in V$, induces a G -isomorphism between V/W^\perp and the dual space W^* . Since V is completely reducible, we may find an irreducible G -submodule $U \cong W^*$ such that the form is non-singular on $X = W \oplus U$. If the form $(\ , \)$ is totally isotropic on W and $n > 2m$, then $V = X \oplus X^\perp$ where X^\perp is not trivial. If the form $(\ , \)$ is non-singular on W , then $V = W \oplus W^\perp$ where W^\perp is not trivial. In both cases we may view $G \lesssim \text{SCRSp}(V_1) \times \text{SCRSp}(V_2)$ as linear groups where $\dim(V_1), \dim(V_2) < n$. If the form $(\ , \)$ is totally isotropic on W and $n = 2m$, then $V = W \oplus U$ and the action of G on V is a pair. We use this fact in the following arguments.

We prove these different cases one by one.

- (1) Let $(n, \mathbb{F}) = (2, \mathbb{F}_2)$. Then $G \lesssim S_3$ and $|G| \mid 6$.
- (2) Let $(n, \mathbb{F}) = (4, \mathbb{F}_2)$. Assume V is irreducible, then the result follows from Lemma 2.9. Assume V is reducible, then $G \lesssim S_3 \times S_3$, $|G| \mid 6^2$ and $G_{\pi_0} = 1$.
- (3) Let $(n, \mathbb{F}) = (6, \mathbb{F}_2)$. Assume V is irreducible and not quasi-primitive, then by Lemma 2.7 G satisfies one of the following:
 - (a) $t = 2$ and $\dim(V_1) = 3$, the action of N on V must be a pair. Thus $N \lesssim \Gamma(2^3)$ and $\mathbf{F}(N) \cong Z_7$. $G_{\pi_0} \cong Z_7$, $|G| \leq 21 \cdot 2 = 42$ and $G_{\pi_0} \subseteq \mathbf{F}(G)$.
 - (b) $t = 3$ and $G \lesssim S_3 \wr S_3$. Thus $|G| \mid 6^4$ and $G_{\pi_0} = 1$.
Assume V is irreducible and quasi-primitive, then G satisfies one of the following:
 - (a) $e = 1$ and $|G| \leq (2^3 + 1) \cdot 6 = 54$ by Lemma 2.8. Clearly $G_{\pi_0} = 1$.
 - (b) $e = 3$. By Theorem 2.2, $A/F \lesssim \text{SL}(2, 3)$, $|W| = 2^2$ and $\dim(W) \mid 2$. Thus $|U| = 3$ and $|G/A| \mid 2$. $|G| \mid 2 \cdot 24 \cdot 3^3$ by Lemma 2.3. Clearly $G_{\pi_0} = 1$.
Assume V is reducible, then G satisfies one of the following:
 - (a) $G \lesssim \text{GL}(3, 2) \times \text{GL}(3, 2)$ and the action of G on V is a pair. Thus $G \lesssim \Gamma(2^3)$, $G_{\pi_0} \cong Z_7$ and $G_{\pi_0} \subseteq \mathbf{F}(G)$. Thus $|G| \leq 21$ and $\text{NEP}_{\{2,3\}'}(G) \leq 6$.

- (b) $G \in \text{SCRSp}(2, 2) \times \text{SCRSp}(4, 2)$. Thus $|G| \leq 6^3 \cdot 2$, $\text{NEP}_{\{2,3\}'}(G) \leq 4$, G_{π_0} is abelian and $G_{\pi_0} \subseteq \mathbf{F}(G)$ by (1) and (2).

Hence the result holds in all cases.

- (4) Let $(n, \mathbb{F}) = (8, \mathbb{F}_2)$. Assume V is irreducible and not quasi-primitive, then by Lemma 2.7 G satisfies one of the following:

- (a) $t = 2$ and $G \lesssim H \wr S_2$ where H is an irreducible subgroup of $\text{GL}(4, 2)$. By Lemma 2.9, $|G| \leq (6^2 \cdot 2)^2 \cdot 2$ and $\text{NEP}_{\{2,3\}'}(G) \leq 5 \cdot 5 - 1 = 24$.
- (b) $t = 4$ and $G \lesssim S_3 \wr S_4$. Clearly $|G| \mid 6^4 \cdot 24$ and $G_{\pi_0} = 1$.

Assume V is irreducible and quasi-primitive. Since $2 \nmid e \mid 8$, $e = 1$. By Lemma 2.8, $|G| \mid (2^4 + 1) \cdot 8 = 136$, $G_{\pi_0} \cong Z_{17}$ and $\text{NEP}_{\{2,3\}'}(G) \leq 16$.

Assume V is reducible, then G satisfies one of the following:

- (a) $G \lesssim \text{GL}(4, 2) \times \text{GL}(4, 2)$ and the action of G on V is a pair. Thus by Lemma 2.9, $|G| \leq 6^2 \cdot 2$ and $\text{NEP}_{\{2,3\}'}(G) \leq 4$.
- (b) $G \in \text{SCRSp}(4, 2) \times \text{SCRSp}(4, 2)$. Thus $|G| \leq 6^4 \cdot 4$, $\text{NEP}_{\{2,3\}'}(G) \leq 5 \cdot 5 - 1 = 24$ by (2).
- (c) $G \in \text{SCRSp}(2, 2) \times \text{SCRSp}(6, 2)$. Thus $|G| \leq 6^5$, $\text{NEP}_{\{2,3\}'}(G) \leq 1 \cdot 7 - 1 = 6$ by (1) and (3).

Hence the result holds in all cases.

- (5) Let $(n, \mathbb{F}) = (2, \mathbb{F}_3)$, then $G \lesssim \text{Sp}(2, 3) \cong \text{SL}(2, 3)$ and $|G| \mid 24$.
- (6) Let $(n, \mathbb{F}) = (4, \mathbb{F}_3)$. Assume V is irreducible and not quasi-primitive, then by Lemma 2.7 G satisfies one of the following:

- (a) $t = 2$ and the form is totally isotropic on V_1 , the action of N on V is a pair. $|N| \mid |\text{GL}(2, 3)| = 48$ and $|G| \mid 96$.
- (b) $t = 2$ and the form is non-singular on V_1 , $G \lesssim \text{SL}(2, 3) \wr S_2$. Thus $|G| \mid 24^2 \cdot 2$ and $G_{\pi_0} = 1$.

Assume further the action of G on V is quasi-primitive. It is well known that a maximal subgroup of $\text{Sp}(4, 3)$ is isomorphic to one of the five groups M_1, M_2, M_3, M_4 and M_5 , where $M_1 \cong \text{SL}(2, 3) \wr S_2$ and $|M_2| = |M_3| = 2^4 \cdot 3^4$, $|O_3(M_2)| = |O_3(M_3)| = 3^3$, $M_4 = 2.S_6$, $M_5 = (D_8 \wr Q_8).A_5$. We can hence assume G is maximal among the solvable subgroups of M_2, M_3, M_4 or M_5 . Assume G is a subgroup of M_2 or M_3 , then clearly $|G| \mid 48$. Assume G is a subgroup of M_4 , it is not hard to show that $|G| \mid 96$ or $|G| \mid 40$. Assume G is a subgroup of M_5 , then $G \cong (D_8 \wr Q_8).L$ where L is S_3 , A_4 or F_{10} and thus we know $|G| \leq 384$ and $|G| \leq 320$ if $5 \mid |G|$. It is checked by GAP [7] that for all G quasi-primitive and $|G| \leq 384$, $\text{NEP}_5(G) \leq 64$ and G will have no elements with other prime order.

Assume V is reducible, then G satisfies one of the following:

- (a) $G \lesssim \text{GL}(2, 3) \times \text{GL}(2, 3)$ and the action of G on V is a pair. Thus $|G| \mid 48$ and $G_{\pi_0} = 1$.
- (b) $G \lesssim \text{SL}(2, 3) \times \text{SL}(2, 3)$. Thus $|G| \mid 24^2$ and $G_{\pi_0} = 1$.

Hence the result holds in all cases.

□

Lemma 2.11. *Let G be a finite solvable group on a finite set Ω . Then there exists a subset $\Delta \subseteq \Omega$ such that $\text{Stab}_G(\Delta)$ is a $\{2, 3\}$ -group. Here, Δ can be chosen to have non-empty intersection with every orbit of G on Ω .*

Proof. This is [19, Corollary 5.7(a)].

□

3. ORBIT THEOREMS

In this section, we prove the key orbit theorem of this paper. The proof relies on some of my previous work. [28] and [29] give a complete list of e such that a solvable quasi-primitive group G will have regular orbits on V . Based on this, we have the following.

Theorem 3.1. *Suppose that a finite solvable group G acts faithfully, irreducibly and quasi-primitively on a finite vector space V . By Theorem 2.2, G will have a uniquely determined normal subgroup E which is a direct product of extra-special p -groups for various p and $e = \sqrt{|E/\mathbf{Z}(E)|}$. Assume $e = 5, 6, 7$ or $e \geq 10$ and $e \neq 16$, then G will have at least two regular orbits on V .*

Proof. This is [28, Theorem 3.1] and [29, Theorem 3.1]. \square

Theorem 3.2. *Let G be a finite solvable group and let V be a finite, faithful, irreducible and quasi-primitive G -module. Then there exists a normal subgroup $K \subseteq \mathbf{F}_2(G)$ and there exist two G -orbits with representatives $v_a, v_b \in V$ such that for any $H \in \text{Hall}_{\pi_0}(G)$, we have $\mathbf{C}_H(v_a)$ and $\mathbf{C}_H(v_b) \subseteq K$. The Hall π_0 -subgroup of $K\mathbf{F}(G)/\mathbf{F}(G)$ and the Hall π_0 -subgroup of $K \cap \mathbf{F}(G)$ are abelian. Furthermore, either $\mathbf{C}_{O_{\pi_0}(K \cap \mathbf{F}(G))}(v_a) = 1$ or $\mathbf{C}_{O_{\pi_0}(K \cap \mathbf{F}(G))}(v_b) = 1$.*

Proof. We adopt the notation in Theorem 2.2. By Theorem 3.1, we may assume $e = 1, 2, 3, 4, 8, 9, 16$. Since e is not divisible by a prime $p \geq 5$, we may assume that $|G/A||A/F|$ is divisible by some prime $p \geq 5$. Otherwise the result is clear since all the elements of π_0 -order that belong to $\mathbf{F}(G)$ act fixed point freely on any nontrivial vectors of V .

In order to show that G has two π_0 -regular orbits on V it suffices to check that

$$\left| \bigcup_{P \in \text{SP}_{\pi_0}(G)} \mathbf{C}_V(P) \right| + |G| < |V|.$$

We will divide the set $\text{SP}_{\pi_0}(G)$ into a union of sets A_i . Clearly $\left| \bigcup_{P \in \text{SP}_{\pi_0}(G)} \mathbf{C}_V(P) \right| \leq \sum_i \left| \bigcup_{P \in A_i} \mathbf{C}_V(P) \right|$. We will find $\beta_i < e$ such that $|\mathbf{C}_V(P)| \leq |W|^{\beta_i b}$ for all $P \in A_i$. We will find a_i such that $|A_i| \leq a_i$. Since $|V| = |W|^{eb}$, it suffices to check that

$$\sum_i a_i \cdot |W|^{\beta_i b} / |W|^{eb} + |G| / |W|^{eb} < 1.$$

We call this inequality \star .

Assume $e = 16$. Define $A_1 = \{\langle x \rangle \mid x \in \text{EP}_{\pi_0}(A \setminus F)\}$. Thus $|\mathbf{C}_V(P)| \leq |W|^{5b}$ for all $P \in A_1$ by Lemma 2.4(2) and we set $\beta_1 = 5$. $|A_1| \leq 24 \cdot 2^8 \cdot (|W| - 1)/2/4 = a_1$ by Lemma 2.5(2) and Lemma 2.10(4). Define $A_2 = \{\langle x \rangle \mid x \in \text{EP}_{\pi_0}(G \setminus A)\}$. Thus $|\mathbf{C}_V(P)| \leq |W|^{8b}$ for all $P \in A_2$ by Lemma 2.4(3) and we set $\beta_2 = 8$. $|A_2| \leq \lfloor \log_5(\dim(W)) \rfloor \cdot 24 \cdot 6^4 \cdot 2^8 \cdot (|W| - 1) = a_2$ by Lemma 2.6. $|G| \leq \dim(W) \cdot 24 \cdot 6^4 \cdot 2^8 \cdot (|W| - 1)$. It is routine to check that \star is satisfied.

Assume $e = 9$.

Assume that $p \mid |G/A|$ for some $p \geq 5$. Thus $p \mid \dim(W)$. Since $p \mid \dim(W)$ and $3 \mid |W| - 1$, $|W| \geq 4^p$. Define $A_1 = \{\langle x \rangle \mid x \in \text{EP}_{\pi_0}(A \setminus F)\}$. Thus $|\mathbf{C}_V(P)| \leq |W|^{3b}$ by Lemma 2.4(2) for all $P \in A_1$ and we set $\beta_1 = 3$. $|A_1| \leq 64 \cdot 3^4 \cdot (|W| - 1)/3/4 = a_1$ by Lemma 2.5(2) and Lemma 2.10(6). Define $A_2 = \{\langle x \rangle \mid x \in \text{EP}_{\pi_0}(G \setminus A)\}$. Thus $|\mathbf{C}_V(P)| \leq |W|^{\frac{9}{5}b}$ for all $P \in A_2$

and we set $\beta_2 = \frac{9}{5}$ by Lemma 2.4(3). $|A_2| \leq \lfloor \log_5(\dim(W)) \rfloor \cdot 24^2 \cdot 2 \cdot 3^4 \cdot (|W| - 1) = a_2$ by Lemma 2.6. $|G| \leq \dim(W) \cdot 24^2 \cdot 2 \cdot 3^4 \cdot (|W| - 1)$. It is routine to check that \star is satisfied.

Assume that $p \nmid |G/A|$ for any $p \geq 5$. Thus we may assume that $p \mid |A/F|$ for some $p \geq 5$. $|A/F| \leq 320$ by Lemma 2.10(6). Define $A_1 = \{\langle x \rangle \mid x \in \text{EP}_{\pi_0}(A \setminus F)\}$. Thus $|\mathbf{C}_V(P)| \leq |W|^{3b}$ for all $P \in A_1$ by Lemma 2.4(2) and we set $\beta_1 = 3$. $|A_1| \leq 64 \cdot 3^4 \cdot (|W| - 1)/3/4 = a_1$ by Lemma 2.5(2) and Lemma 2.10(6). $|G| \leq \dim(W) \cdot 320 \cdot 3^4 \cdot (|W| - 1)$. It is routine to check that \star is satisfied.

Assume $e = 8$. Since $(A/F)_{\pi_0}$ is abelian and $(A/F)_{\pi_0} \subseteq \mathbf{F}(A/F)$, we may assume that $p \mid |G/A|$ for some $p \geq 5$. Thus $p \mid \dim(W)$. Since $p \mid \dim(W)$ and $2 \mid |W| - 1$, $|W| \geq 3^p$. Define $A_1 = \{\langle x \rangle \mid x \in \text{EP}_{\pi_0}(A \setminus F)\}$. Thus $|\mathbf{C}_V(P)| \leq |W|^{2b}$ for all $P \in A_1$ by Lemma 2.4(2) and we set $\beta_1 = 2$. $|A_1| \leq 6 \cdot 2^6 \cdot (|W| - 1)/2/4 = a_1$ by Lemma 2.5(2) and Lemma 2.10(3). Define $A_2 = \{\langle x \rangle \mid x \in \text{EP}_{\pi_0}(G \setminus A)\}$. Thus $|\mathbf{C}_V(P)| \leq |W|^{\frac{8}{5}b}$ for all $P \in A_2$ by Lemma 2.4(3) and we set $\beta_2 = \frac{8}{5}$. $|A_2| \leq \lfloor \log_5(\dim(W)) \rfloor \cdot 6^4 \cdot 2^6 \cdot (|W| - 1) = a_2$ by Lemma 2.6. $|G| \leq \dim W \cdot 6^4 \cdot 2^6 \cdot (|W| - 1)$. It is routine to check that \star is satisfied.

Assume $e = 4$. Since $(A/F)_{\pi_0}$ is abelian and $(A/F)_{\pi_0} \subseteq \mathbf{F}(A/F)$, we may assume that $p \mid |G/A|$ for some $p \geq 5$. Thus $p \mid \dim(W)$. Since $p \mid \dim(W)$ and $2 \mid |W| - 1$, $|W| \geq 3^p$. Define $A_1 = \{\langle x \rangle \mid x \in \text{EP}_{\pi_0}(A \setminus F)\}$. $|\mathbf{C}_V(P)| \leq |W|^b$ for all $P \in A_1$ by Lemma 2.4(2) and we set $\beta_1 = 1$. $|A_1| \leq 4 \cdot 2^4 \cdot (|W| - 1)/2/4 = a_1$ by Lemma 2.5(2) and Lemma 2.10(2). Define $A_2 = \{\langle x \rangle \mid x \in \text{EP}_{\pi_0}(G \setminus A)\}$. Thus $|\mathbf{C}_V(P)| \leq |W|^{\frac{4}{5}b}$ for all $P \in A_2$ by Lemma 2.4(3) and we set $\beta_2 = \frac{4}{5}$. $|A_2| \leq \lfloor \log_5(\dim(W)) \rfloor \cdot 6^2 \cdot 2 \cdot 2^4 \cdot (|W| - 1) = a_2$ by Lemma 2.6. $|G| \leq \dim W \cdot 6^2 \cdot 2 \cdot 2^4 \cdot (|W| - 1)$. It is routine to check that \star is satisfied.

Assume $e = 3$. Since $(A/F)_{\pi_0} = 1$, we may assume that $p \mid |G/A|$ for some $p \geq 5$. Thus $p \mid \dim(W)$. Since $p \mid \dim(W)$ and $3 \mid |W| - 1$, $|W| \geq 4^p$. Define $A_1 = \{\langle x \rangle \mid x \in \text{EP}_{\pi_0}(G \setminus A)\}$. Thus $|\mathbf{C}_V(P)| \leq |W|^{\frac{3}{5}b}$ for all $P \in A_1$ by Lemma 2.4(3) and we set $\beta_1 = \frac{3}{5}$. $|A_1| \leq \lfloor \log_5(\dim(W)) \rfloor \cdot 24 \cdot 3^2 \cdot (|W| - 1) = a_1$ by Lemma 2.6. $|G| \leq \dim W \cdot 24 \cdot 9 \cdot (|W| - 1)$. It is routine to check that \star is satisfied.

Assume $e = 2$. Since $(A/F)_{\pi_0} = 1$, we may assume that $p \mid |G/A|$ for some $p \geq 5$. Thus $p \mid \dim(W)$. Since $p \mid \dim(W)$ and $2 \mid |W| - 1$, $|W| \geq 3^p$. Define $A_1 = \{\langle x \rangle \mid x \in \text{EP}_{\pi_0}(G \setminus A)\}$. Thus $|\mathbf{C}_V(P)| \leq |W|^{\frac{2}{5}b}$ for all $P \in A_1$ by Lemma 2.4(3) and we set $\beta_1 = \frac{2}{5}$. Since $\text{Aut}(S_3) \cong S_3$ and $\text{Aut}(Z_3) \cong Z_2$, all the elements of prime order $p \geq 5$ in $G \setminus A$ acts trivially on A/F . Since G/A is cyclic, $\text{NSP}_{\pi_0}(G/F) \leq \log_5(\dim(W))$. It is easy to see that $|A_1| \leq \lfloor \log_5(\dim(W)) \rfloor \cdot 2^2 \cdot (|W| - 1) = a_1$. $|G| \leq \dim W \cdot 6 \cdot 4 \cdot (|W| - 1)$. It is routine to check that \star is satisfied.

Assume $e = 1$, we have $G \leq \Gamma(V)$. We know that $\text{fl}(G) \leq 2$, $(\mathbf{F}_2(G)/\mathbf{F}(G))_{\pi_0}$ and $\mathbf{F}(G)_{\pi_0}$ are abelian. \square

We now restate Theorem A for convenience. This theorem may be viewed as the linear group analog of Lemma 2.11.

Theorem A. *Let G be a finite solvable group and let V be a finite, faithful and completely reducible G -module (possibly of mixed characteristic). Then there exists $K \triangleleft G$, $K \subseteq \mathbf{F}_2(G)$*

and there exist two G -orbits with representatives $v_a, v_b \in V$ such that for any $H \in \text{Hall}_{\pi_0}(G)$, we have $\mathbf{C}_H(v_a) \subseteq K$ and $\mathbf{C}_H(v_b) \subseteq K$. The π_0 -subgroup of $K\mathbf{F}(G)/\mathbf{F}(G)$ and the π_0 -subgroup of $K \cap \mathbf{F}(G)$ are abelian. Furthermore, $\mathbf{C}_{O_{\pi_0}(K \cap \mathbf{F}(G))}(v_a) \cap \mathbf{C}_{O_{\pi_0}(K \cap \mathbf{F}(G))}(v_b) = 1$.

Proof. We consider minimal counterexample on $|G| + \dim V$.

Step 1. V is an irreducible G -module. Assume not, we have $V = V_1 \oplus V_2$ and each V_i is a non-trivial G -module. Let $C_i = \mathbf{C}_G(V_i)$ and V_i is a faithful G/C_i -module. Let $G_i = G/C_i$ and we know that $G \lesssim G_1 \times G_2$. There exists $v_{ia}, v_{ib} \in V_i$ where v_{ia}, v_{ib} are in different G_i orbits and $K_i \triangleleft G_i$ such that for any $H_i \in \text{Hall}_{\pi_0}(G_i)$, $\mathbf{C}_{H_i}(v_{ia}) \subseteq K_i \subseteq \mathbf{F}_2(G_i)$ and $\mathbf{C}_{H_i}(v_{ib}) \subseteq K_i \subseteq \mathbf{F}_2(G_i)$. Also the π_0 -subgroup of $K_i\mathbf{F}(G_i)/\mathbf{F}(G_i)$ and the π_0 -subgroup of $K_i \cap \mathbf{F}(G_i)$ are abelian and $\mathbf{C}_{H_i \cap \mathbf{F}(G)}(v_{ia}) \cap \mathbf{C}_{H_i \cap \mathbf{F}(G)}(v_{ib}) = 1$.

Let $v_a = v_{1a} + v_{2a}$, $v_b = v_{1b} + v_{2b}$ and $K = G \cap (K_1 \times K_2)$. Let $H \in \text{Hall}_{\pi_0}(G)$ and $H_i = HC_i/C_i$. $\mathbf{C}_H(v_a) \subseteq \mathbf{C}_{H_1}(v_{1a}) \times \mathbf{C}_{H_2}(v_{2a}) \subseteq K_1 \times K_2$. $\mathbf{C}_H(v_b) \subseteq \mathbf{C}_{H_1}(v_{1b}) \times \mathbf{C}_{H_2}(v_{2b}) \subseteq K_1 \times K_2$. Clearly $K\mathbf{F}(G)/\mathbf{F}(G) \subseteq K_1\mathbf{F}(G_1)/\mathbf{F}(G_1) \times K_2\mathbf{F}(G_2)/\mathbf{F}(G_2)$ and $K \cap \mathbf{F}(G) \subseteq K_1 \cap \mathbf{F}(G_1) \times K_2 \cap \mathbf{F}(G_2)$.

$\mathbf{C}_{O_{\pi_0}(K \cap \mathbf{F}(G))}(v_{1a}+v_{2a}) \cap \mathbf{C}_{O_{\pi_0}(K \cap \mathbf{F}(G))}(v_{1b}+v_{2b}) \subseteq (\mathbf{C}_{O_{\pi_0}(K_1 \cap \mathbf{F}(G_1))}(v_{1a}) \cap \mathbf{C}_{O_{\pi_0}(K_1 \cap \mathbf{F}(G_1))}(v_{1b})) \times (\mathbf{C}_{O_{\pi_0}(K_2 \cap \mathbf{F}(G_2))}(v_{2a}) \cap \mathbf{C}_{O_{\pi_0}(K_2 \cap \mathbf{F}(G_2))}(v_{2b})) = 1$.

Step 2. If V is not quasi-primitive and there exists a normal subgroup N of G such that $V_N = V_1 \oplus \dots \oplus V_m$ for $m > 1$ homogeneous components V_i of V_N . If N is maximal with this property, then $S = G/N$ primitively permutes the V_i . Also $V = V_1^G$, induced from $\mathbf{N}_G(V_1)$. Let $L_1 = \mathbf{N}_G(V_1)/\mathbf{C}_G(V_1)$, then L_1 acts faithfully and irreducibly on V_1 and G is isomorphic to a subgroup of $L_1 \wr S$. Now, by induction, L_1 has at least two orbits of elements with representatives $v_1, u_1 \in V_1$ and $K_1 \triangleleft L_1$ such that for $H_1 \in \text{Hall}_{\pi_0}(\mathbf{C}_{L_1}(v_1))$ or $H_1 \in \text{Hall}_{\pi_0}(\mathbf{C}_{L_1}(u_1))$ we have $H_1 \subseteq K_1 \subseteq \mathbf{F}_2(L_1)$. Also the π_0 -subgroup of $K_1\mathbf{F}(L_1)/\mathbf{F}(L_1)$ and the π_0 -subgroup of $K_1 \cap \mathbf{F}(L_1)$ are abelian. Also $\mathbf{C}_{O_{\pi_0}(K_1 \cap \mathbf{F}(L_1))}(v_1) \cap \mathbf{C}_{O_{\pi_0}(K_1 \cap \mathbf{F}(L_1))}(u_1) = 1$. By Lemma 2.11, there exist two G -orbits with representatives $v_a, v_b \in V$ and $K \triangleleft G$ such that for $H \in \text{Hall}_{\pi_0}(\mathbf{C}_G(v_a))$ or $H \in \text{Hall}_{\pi_0}(\mathbf{C}_G(v_b))$ we have $H \subseteq K \subseteq \mathbf{F}_2(G)$. Also the π_0 -subgroup of $K\mathbf{F}(G)/\mathbf{F}(G)$ and the π_0 -subgroup of $K \cap \mathbf{F}(G)$ are abelian. Furthermore, $\mathbf{C}_{O_{\pi_0}(K \cap \mathbf{F}(G))}(v_a) \cap \mathbf{C}_{O_{\pi_0}(K \cap \mathbf{F}(G))}(v_b) = 1$.

Step 3. V is quasi-primitive, the claim follows by Theorem 3.2. Final contradiction. \square

Theorem 3.3. *Let G be a finite solvable group and let V be a finite, faithful and completely reducible G -module (possibly of mixed characteristic). Let p be a prime and $p \geq 5$. Then there exists $K \triangleleft G$, $K \subseteq \mathbf{F}_2(G)$ and there exists a G -orbit with representative $v \in V$ such that for any $P \in \text{Syl}_p(G)$ we have $\mathbf{C}_P(v) \subseteq K$. Also the p -subgroup of $K\mathbf{F}(G)/\mathbf{F}(G)$ and the p -subgroup of $K \cap \mathbf{F}(G)$ are abelian. Furthermore, $|\mathbf{C}_{O_p(K \cap \mathbf{F}(G))}(v)| \leq |O_p(K \cap \mathbf{F}(G))|^{1/2}$.*

Proof. By Theorem A, there exist two G -orbits with representatives $v_a, v_b \in V$ and $K \triangleleft G$ such that for $H \in \text{Hall}_{\pi_0}(G)$, we have $\mathbf{C}_H(v_a) \subseteq K \subseteq \mathbf{F}_2(G)$ and $\mathbf{C}_H(v_b) \subseteq K \subseteq \mathbf{F}_2(G)$. The π_0 -subgroups of $K\mathbf{F}(G)/\mathbf{F}(G)$ and $K \cap \mathbf{F}(G)$ are abelian. Furthermore $\mathbf{C}_{O_{\pi_0}(K \cap \mathbf{F}(G))}(v_a) \cap \mathbf{C}_{O_{\pi_0}(K \cap \mathbf{F}(G))}(v_b) = 1$. Let $P \in \text{Syl}_p(G)$, then $P \subseteq H \in \text{Hall}_{\pi_0}(G)$ for some H . Thus $\mathbf{C}_P(v_a) \subseteq \mathbf{C}_H(v_a) \subseteq K \subseteq \mathbf{F}_2(G)$ and $\mathbf{C}_P(v_b) \subseteq \mathbf{C}_H(v_b) \subseteq K \subseteq \mathbf{F}_2(G)$. Also the p -subgroups of $K\mathbf{F}(G)/\mathbf{F}(G)$ and $K \cap \mathbf{F}(G)$ are abelian.

Let $P_1 = O_p(K \cap \mathbf{F}(G))$, then $\mathbf{C}_{P_1}(v_a) \cap \mathbf{C}_{P_1}(v_b) = 1$. Since $|\mathbf{C}_{P_1}(v_a)| \cdot |\mathbf{C}_{P_1}(v_b)| = \frac{|\mathbf{C}_{P_1}(v_a)| \cdot |\mathbf{C}_{P_1}(v_b)|}{|\mathbf{C}_{P_1}(v_a) \cap \mathbf{C}_{P_1}(v_b)|} = |\mathbf{C}_{P_1}(v_a)\mathbf{C}_{P_1}(v_b)| \leq |P_1|$. It follows that, either $|\mathbf{C}_{P_1}(v_a)| \leq \sqrt{|P_1|}$ or $|\mathbf{C}_{P_1}(v_b)| \leq \sqrt{|P_1|}$. \square

4. BLOCKS OF SMALL DEFECT

Let G be a finite group. Let p be a prime and $|G|_p = p^n$. An irreducible ordinary character of G is called p -defect 0 if and only if its degree is divisible by p^n . By [6, Theorem 4.18], G has a character of p -defect 0 if and only if G has a p -block of defect 0. An important question in the modular representation theory of finite groups is to find the group-theoretic conditions for the existence of characters of p -defect 0 in a finite group. It is an interesting problem to give necessary and sufficient conditions for the existence of p -blocks of defect zero. If a finite group G has a character of p -defect 0, then $O_p(G) = 1$ [6, Corollary 6.9]. Unfortunately, the converse is not true. Zhang [31] and Hiroshi [11, 12] provided various sufficient conditions where a finite group G has a block of defect zero.

Although the block of defect zero may not exist in general, one could try to find the smallest defect $d(B)$ of a block B of G . One of the results along this line is given by [5, Theorem A]. In [5], Espuelas and Navarro bounded the smallest defect $d(B)$ of a block B of G using the p -part of G . Using an orbit theorem [4, Theorem 3.1] of solvable linear groups of odd order, they showed the following result. Let G be a (solvable) group of odd order such that $O_p(G) = 1$ and $|G|_p = p^n$, then G contains a p -block B such that $d(B) \leq \lfloor n/2 \rfloor$. The bound is best possible, as shown by an example in [5].

It is not true in general that there exists a block B with $d(B) \leq \lfloor n/2 \rfloor$, as $G = A_7(p = 2)$ shows us. However, the counterexamples were only found for $p = 2$ and $p = 3$. By work of Michler and Willems [17, 26] every simple group except possibly the alternating group has a block of defect zero for $p \geq 5$. The alternating group case was settled by Granville and Ono in [10] using number theory. In fact, they proved the t -core partition conjecture and the most difficult case was handled by modular forms. Based on this, the following question raised by Espuelas and Navarro [5] seems to be natural. If G is a finite group with $O_p(G) = 1$, $p \geq 5$, and $|G|_p = p^n$, does G contain a block of defect less than $\lfloor \frac{n}{2} \rfloor$?

In this section, we study this question and show that for solvable group G , $O_p(G) = 1$ and $p \geq 5$, G contains a block of defect less than or equal to $\lfloor \frac{3n}{5} \rfloor$. The proof relies on the previous orbit theorem (Theorem 3.3). The bound we obtain here is pretty sharp since $\lfloor \frac{n}{2} \rfloor$ is the best one may get. We restate Theorem B for convenience.

Theorem B. *Let G be a finite solvable group such that $O_p(G) = 1$ for $p \geq 5$ and let $|G|_p = p^n$. Then G contains a p -block B such that $d(B) \leq \lfloor \frac{3n}{5} \rfloor$.*

Proof. Induction on $|G|$. Consider $\tilde{G} = G/\Phi(G)$. As $\mathbf{F}(G/\Phi(G)) = \mathbf{F}(G)/\Phi(G)$, we have that $O_p(\tilde{G}) = 1$ and $|\tilde{G}|_p = |G|_p$. If $\Phi(G) \neq 1$, then the result is true for \tilde{G} . Let \tilde{B} be a p -block of \tilde{G} such that $d(\tilde{B}) < \lfloor \frac{3n}{5} \rfloor$. By [6, Lemma V.4.3], there exists a p -block B of G such that $d(B) = d(\tilde{B})$. Hence we may assume that $\Phi(G) = 1$.

Now, $V = \text{Irr}(\mathbf{F}(G))$ is a faithful and completely reducible $\bar{G} = G/\mathbf{F}(G)$ -module (over different fields, possibly). Put $V = V_1 \oplus \cdots \oplus V_t$, where each V_i is an irreducible \bar{G} -module. Define $K_i = \mathbf{C}_{\bar{G}}(V_i)$ and $G_i = \bar{G}/K_i$. By Theorem 3.3, there exists a normal subgroup $K \subseteq \mathbf{F}_2(\bar{G})$ and there exists a \bar{G} -orbit with representatives $\lambda \in V$ such that for any $P \in \text{Syl}_p(\bar{G})$ we have $\mathbf{C}_P(\lambda) \subseteq K$. Also the p -subgroup of $K\mathbf{F}(\bar{G})/\mathbf{F}(\bar{G})$ and the p -subgroup of $K \cap \mathbf{F}(\bar{G})$ are abelian. Furthermore, $|\mathbf{C}_{O_p(K \cap \mathbf{F}(\bar{G}))}(\lambda)| \leq |O_p(K \cap \mathbf{F}(\bar{G}))|^{1/2}$. Let $p^n = |\bar{G}|_p$, $p^{n_1} = |K \cap \mathbf{F}(\bar{G})|_p$, $p^{n_2} = |K\mathbf{F}(\bar{G}) : \mathbf{F}(\bar{G})|_p$ and $p^{n_3} = |\bar{G} : K|_p$. Clearly $n = n_1 + n_2 + n_3$.

Take $\chi \in \text{Irr}(G)$ lying over λ and let B be the p -block of G containing χ . As $\mathbf{F}(G)$ is a p' -group, [6, Lemma V.2.3] shows that every irreducible character ψ in B has λ as an irreducible constituent. Now $\psi(1)_p \geq |\bar{G} : \mathbf{C}_{\bar{G}}(\lambda)|_p \cdot |K \cap \mathbf{F}(\bar{G})|_p^{1/2} \geq p^{n_3} \cdot p^{n_1/2}$ by Theorem 3.3.

Let $P/\mathbf{F}(\bar{G})$ be the Sylow p -subgroup of $K\mathbf{F}(\bar{G})/\mathbf{F}(\bar{G})$ and let P be the preimage of it. Let $Y = O_{p'}(\mathbf{F}(\bar{G}))$, observe that $W = \text{Irr}(Y/\Phi(Y))$ is a faithful and completely reducible $P/\mathbf{F}(\bar{G})$ -module. Since $P/\mathbf{F}(\bar{G})$ is abelian, there exists $\mu \in W$ such that $C_P(\mu) = \mathbf{F}(\bar{G})$. We may view μ as a character of the preimage X of Y in G . Observe that X is a p' -group. Take $\xi \in \text{Irr}(G)$ lying over μ . Now ξ lies over an irreducible character ϕ of P lying over μ . Clearly, $\phi(1)_p \geq |P/\mathbf{F}(\bar{G})| = p^{n_2}$. As P is normal in G , we have $\xi(1)_p \geq \phi(1)_p \geq p^{n_2}$. Let B be the p -block of G containing ξ . As X is a p' -group, [6, Lemma V.2.3] shows that every irreducible character δ in B has μ as an irreducible constituent and $\delta(1)_p \geq p^{n_2}$.

Let $P_1/\mathbf{F}(G)$ be the Sylow p -subgroup of $K \cap \mathbf{F}(\bar{G})$ and let P_1 be the preimage of it. Since $P_1/\mathbf{F}(G)$ is normal in $G/\mathbf{F}(G)$, $V = \text{Irr}(\mathbf{F}(G))$ is a faithful and completely reducible $P_1/\mathbf{F}(G)$ -module. Since $P_1/\mathbf{F}(G)$ is abelian, using a similarly argument as the previous paragraph, we may find a block B such that every irreducible character φ in B satisfies $\varphi(1)_p \geq |K \cap \mathbf{F}(\bar{G})|_p = p^{n_1}$.

We know there is a block B such that for every irreducible character α in B , $\alpha(1)_p \geq \max(p^{n_3} \cdot p^{n_1/2}, p^{n_2}, p^{n_1})$. It is not hard to see that $\alpha(1)_p \geq p^{\frac{2n}{5}}$ and thus $d(B) \leq \lfloor \frac{3n}{5} \rfloor$. \square

Remark: Although the result for the solvable group case is satisfactory, the conjecture of Espuelas and Navarro for arbitrary finite groups is wide open.

5. p PART OF $|G : \mathbf{F}(G)|$ AND IRREDUCIBLE CHARACTER DEGREES

If P is a Sylow p -subgroup of a finite group G it is reasonable to expect that the degrees of irreducible characters of G somehow restrict those of P . Let p^a denote the largest power of p dividing $\chi(1)$ for an irreducible character χ of G and $b(P)$ denote the largest degree of an irreducible character of P . Conjecture 4 of Moret o [20] suggested $\log b(P)$ is bounded as a function of a . Moret o and Wolf [22] have proven this for G solvable and even something a bit stronger, namely the logarithm to the base of p of the p -part of $|G : \mathbf{F}(G)|$ is bounded in terms of a . In fact, they showed that $|G : \mathbf{F}(G)|_p \leq p^{19a}$. Moret o and Wolf [22] also proved that $|G : \mathbf{F}(G)|_p \leq p^{2a}$ for odd order groups, this can also be deduced from [5]. This bound is best possible, as shown by an example in [5]. It is possible that $p^a < b(P)$ at least when $p = 2$, as shown by an example of Isaacs [20, Example 5.1].

Moret o and Wolf [22] suggested that a better bound $|G : \mathbf{F}(G)|_p \leq p^{2a}$ might be true for all solvable groups. In fact, they believe [22, Question 2.2] that for solvable groups one may find two irreducible characters χ_1 and χ_2 such that $|G : \mathbf{F}(G)| \mid \chi_1(1)\chi_2(1)$. Although it is difficult to answer this question in general, we are able to prove a closely related result using the previous orbit theorem. As a corollary, we show that $|G : \mathbf{F}(G)|_p \leq p^{3a}$ for $p \geq 5$.

The following result is closely related to [22, Theorem A].

Theorem 5.1. *If G is solvable, there exists a product $\theta = \chi_1\chi_2\chi_3$ of distinct irreducible characters χ_1 , χ_2 and χ_3 such that $|G : \mathbf{F}(G)|_{\pi_0}$ divides $\theta(1)$.*

Proof. By Theorem A, we may choose $\chi \in \text{Irr}(G)$ and $K \triangleleft G$ such that $\mathbf{F}(G)$ is not in $\text{Ker}\chi$ and $|G : K|_{\pi_0}$ divides $\chi(1)$. We can choose $\phi \in \text{Irr}(G)$ such that $\mathbf{F}(G)$ is in $\text{Ker}\phi$ and $|K\mathbf{F}_2(G) : \mathbf{F}_2(G)|_{\pi_0}$ divides $\phi(1)$. We can choose $\mu \in \text{Irr}(G)$ such that $\mathbf{F}(G)$ is not in $\text{Ker}\mu$ and $|K \cap \mathbf{F}(G) : \mathbf{F}(G)|_{\pi_0}$ divides $\mu(1)$.

Then ϕ is distinct since $\mathbf{F}(G)$ is in $\text{Ker}\phi$ but not in $\text{Ker}\chi$ and $\text{Ker}\mu$. If μ is χ , the product $\theta = \chi\phi$ satisfies the conclusion. Else $\theta = \chi\phi\mu$ does. \square

The following result is closely related to [22, Theorem A'].

Theorem 5.2. *If G is solvable, there exist conjugacy classes C_1, C_2 and C_3 such that $|G : \mathbf{F}(G)|_{\pi_0}$ divides $|C_1||C_2||C_3|$.*

The following result improves [22, Corollary B] for $p \geq 5$.

Corollary 5.3. *Suppose that p^{a+1} does not divide $\chi(1)$ for all $\chi \in \text{Irr}(G)$ and let $P \in \text{Syl}_p(G)$ where $p \geq 5$. If G is solvable, then $|G : \mathbf{F}(G)|_p \leq p^{3a}$, $b(P) \leq p^{4a}$ and $\text{dl}(P) \leq \log_2 a + 7$.*

Proof. There exists a product $\theta = \chi_1\chi_2\chi_3$ of distinct irreducible characters χ_i such that $|G : \mathbf{F}(G)|_{\pi_0}$ divides $\theta(1)$ by Theorem 5.1 and so $|G : \mathbf{F}(G)|_p \leq p^{3a}$. If $P \in \text{Syl}_p(G)$, then $b(P) \leq |P : O_p(G)||b(O_p(G))| = |G : \mathbf{F}(G)|_p|b(O_p(G))| \leq p^{3a}p^a = p^{4a}$.

Now, we want to prove the last part of the statement. By [16, Theorem 12.26] and the nilpotency of P , we have that P has an abelian subgroup B of index at most $b(P)^4$. By [23, Theorem 5.1], we deduce that P has a normal abelian subgroup A of index at most $|P : B|^2$. Thus, $|P : A| \leq |P : B| \leq b(P)^{8s}$, where $b(P) = p^s$. By [13, Satz III.2.12], $\text{dl}(P/A) \leq 1 + \log_2(8s)$ and so $\text{dl}(P) \leq 2 + \log_2(8s) = 5 + \log_2(s)$. Since s is at most $4a$, the result follows. \square

We now state the conjugacy analogs of Theorem 5.3. Given a group G , we write $b^*(G)$ to denote the largest size of the conjugacy classes of G . The following result improves [22, Corollary B'] for $p \geq 5$.

Corollary 5.4. *Suppose that p^{a+1} does not divide $|C|$ for all $C \in \text{cl}(G)$ and let $P \in \text{Syl}_p(G)$ where $p \geq 5$. If G is solvable, then $|G : \mathbf{F}(G)|_p \leq p^{3a}$, $b^*(P) \leq p^{4a}$ and $|P'| \leq p^{2a(4a+1)}$.*

Proof. The first statement follows directly from Theorem 5.2. Write $N = O_p(G)$. It is clear that $|N : \mathbf{C}_N(x)|$ divides $|G : \mathbf{C}_G(x)|$ for all $x \in G$. Thus, if we take $x \in P$ we have that

$$|\text{cl}_P(x)| = |P : \mathbf{C}_P(x)| \leq |P : N||N : \mathbf{C}_N(x)| \leq p^{3a}p^a = p^{4a}$$

Finally, to obtain the bounds for the order of P' it suffices to apply a theorem of Vaughan-Lee [14, Theorem VIII.9.12]. \square

6. HUPPERT $\rho - \sigma$ CONJECTURES

In this section we discuss Huppert's $\rho - \sigma$ conjectures.

If n is a positive integer, we denote by $\pi(n)$ the set of all prime divisors of n . Let χ be an irreducible complex character of a group G , we denote by $\pi(\chi)$ the set of all prime divisors of the degree $\chi(1)$ of χ . We define

$$\sigma(G) = \max\{|\pi(\chi)| : \chi \in \text{Irr}(G)\} \text{ and } \rho(G) = \bigcup_{\chi \in \text{Irr}(G)} \pi(\chi).$$

Thus $\rho(G)$ are those primes that divide the degree of some irreducible character of G and $\sigma(G)$ is the maximum number of primes dividing the degree of an irreducible character of G . By Ito's theorem, $\rho(G)$ is precisely the set of all primes p such that G does not have a normal abelian Sylow p -subgroup.

Similarly, if $g \in G$, we denote by $\pi(g^G)$ the set of all prime divisors of $|G : \mathbf{C}_G(g)|$, the size of the conjugacy class of $g \in G$. We define

$$\sigma^*(G) = \max\{|\pi(g^G)| : g \in G\} \text{ and } \rho^*(G) = \bigcup_{g \in G} \pi(g^G).$$

Thus $\rho^*(G)$ is the set of all prime divisors of the sizes of conjugacy classes of G . It is an elementary fact that $\rho^*(G) = \pi(G/\mathbf{Z}(G))$. $\sigma^*(G)$ is the maximum number of distinct primes dividing the order of some conjugacy class of G .

A lot of research has been made on character degrees of finite groups since the eighties due to the interest of B. Huppert. One of the main problems that Huppert raised was his well-known $\rho - \sigma$ conjectures. The Huppert's $\rho - \sigma$ conjectures state that there is an irreducible character χ of G and a conjugacy class C of G such that the degree of χ and $|C|$ are each divisible by many primes. Huppert's $\rho - \sigma$ conjecture is a problem of central importance in group theory and character theory; many people are devoted to the study of this problem.

For the character theoretic $\rho - \sigma$ problem, Huppert conjectured that $|\rho(G)|$ can be bounded in terms of $\sigma(G)$, and if G is solvable, then even $|\rho(G)| \leq 2\sigma(G)$. In [21] Moreto proved that, for any group G , $|\rho(G)| \leq 4\sigma(G)^2 + 6.5\sigma(G) + 13$. This bound was improved to $|\rho(G)| \leq 7\sigma(G)$ by Casolo and Dolfi [3, Theorem 1]. Up to now the best known bound for G solvable is $|\rho(G)| \leq 3\sigma(G) + 2$ and even $|\rho(G)| \leq 3\sigma(G)$ for $|G|$ odd by Manz and Wolf [18, Theorems 1.4 and 1.5].

For the conjugacy class $\rho^* - \sigma^*$ problem, Huppert also conjectured that $|\rho^*(G)| \leq 2\sigma^*(G)$ for G solvable. Casolo [1] showed that $|\rho^*(G)| \leq 2\sigma^*(G)$ for a very large family of groups. But Casolo and Dolfi [2] disproved the obvious conjecture by constructing solvable groups G_n for which $|\rho^*(G_n)|/\sigma^*(G_n) \rightarrow 3$ as $n \rightarrow \infty$. In [21] Moreto proved that, for any group G , $|\rho(G)| \leq 3\sigma^*(G)^2 + 7.5\sigma^*(G) + 11$. This bound was improved to $|\rho(G)| \leq 7\sigma(G)$ by Casolo and Dolfi [3, Theorem 2]. Up to now the best known bound for G solvable is $|\rho^*(G)| \leq 4\sigma^*(G)$ by Zhang [32].

Theorem A yields linear bounds for arbitrary solvable groups in both versions of the problem. Thus, it provides a unified approach to the character-theoretic and the conjugacy class version of the $\rho - \sigma$ conjectures. The following theorem is about the character-theoretic version of the $\rho - \sigma$ conjectures.

Theorem 6.1. *Suppose that M is a normal elementary abelian subgroup of the solvable group G . Assume that $M = \mathbf{C}_G(M)$ is a completely reducible G -module (possibly of mixed characteristic). Set $V = \text{Irr}(M)$ and write $V = V_1 \oplus \cdots \oplus V_m$ for irreducible G -modules V_i . For each i , write $V = Y_i^G$ for primitive modules Y_i . Then there exists $\text{Irr}(G)$ whose degree is divisible by at least half the primes of $\pi_0(G/M)$.*

Proof. We may write each V_i as a direct sum of the G -conjugates of Y_i , $i = 1, \dots, m$. Consequently, $V = X_1 \oplus \cdots \oplus X_n$ for subspaces X_i of V permuted by G (not necessarily transitively) with $\{Y_1, \dots, Y_m\} \subseteq \{X_1, \dots, X_n\}$. Furthermore, if $N_i = \mathbf{N}_G(X_i)$, $C_i = \mathbf{C}_G(X_i)$ and $F_i/C_i = \mathbf{F}(N_i/C_i)$, then X_i is a primitive, faithful N_i/C_i -module. We denote by $\overline{N}_i = N_i/C_i$.

Let $N = \bigcap_i N_i \triangleleft G$ be the kernel of the permutation representation of G on $\{X_1, \dots, X_n\}$. Since $\bigcap_i C_i = M$, we have $\bigcap_i F_i/M = \mathbf{F}(N/M) \triangleleft G/M$. Let $F = \bigcap_i F_i$, so that $F/M = \mathbf{F}(N/M)$.

By Lemma 2.11, we may choose $\Delta \in \{X_1, \dots, X_n\}$ such that $\text{stab}_G(\Delta)/N$ is a $\{2, 3\}$ -group. Furthermore, we can assume that Δ intersects each G -orbit non-trivially. Without loss of generality, $\Delta = \{X_1, \dots, X_l\}$ for some $l \in \{1, \dots, n\}$.

Let $\Delta_{i1} = Y_i^G \cap \Delta$ and $\Delta_{i2} = Y_i^G \setminus \Delta_{i1}$ where $Y_i \in \{Y_1, \dots, Y_m\}$.

Thus, for $j \in \Delta_{i1}$, we may choose non-principle $\lambda_j = \lambda_{ia}^g \in X_j$ such that there exists a normal subgroup $\overline{K}_j \subseteq \mathbf{F}_2(\overline{N}_j)$ and for any $H \in \text{Hall}_{\pi_0}(\overline{N}_j)$, we have $\mathbf{C}_H(\lambda_j) \subseteq \overline{K}_j$ by Theorem 3.2. The π_0 -subgroups of $(\overline{K}_j \cap \mathbf{F}_2(\overline{N}_j))\mathbf{F}(\overline{N}_j)/\mathbf{F}(\overline{N}_j)$ are abelian. For $j \in \Delta_{i2}$, we may choose $\lambda_j = \lambda_{ib}^g \in X_j$ such that there exists a normal subgroup $\overline{K}_j \subseteq \mathbf{F}_2(\overline{N}_j)$ and for any $H \in \text{Hall}_{\pi_0}(\overline{N}_j)$, we have $\mathbf{C}_H(\lambda_j) \subseteq \overline{K}_j$. The π_0 -subgroups of $(\overline{K}_j \cap \mathbf{F}_2(\overline{N}_j))\mathbf{F}(\overline{N}_j)/\mathbf{F}(\overline{N}_j)$ are abelian. Here λ_{ia} and λ_{ib} belong to different N_i orbits.

We define $\lambda = \lambda_1 \dots \lambda_n \in V$.

Finally suppose that $Q \in \text{Syl}_q(G)$ for a prime $q \geq 5$, and Q centralizes λ . Thus $Q \subseteq \text{stab}_G(\Delta)$. But $\text{stab}_G(\Delta)/N$ is a $\{2, 3\}$ -group. Thus $Q \subseteq N$ and we know that $Q \subseteq K = \bigcap K_j \subseteq \mathbf{F}_2(N/M)$ and the π_0 -subgroup of KF/F is abelian.

Since λ_i is non-principle for $i = 1, \dots, l$, λ_i is not centralized by a non-trivial Sylow q -subgroup of $F_i \cap N/C_i \cap N$ by Theorem 2.2(6). Since $Q \cap F_i \in \text{Syl}_q(F_i \cap N)$, we have that $q \nmid |F_i \cap N/C_i \cap N|$ for $i = 1, \dots, l$. By our choice of Δ , each F_j/C_j ($j = 1, \dots, n$) is conjugate to some F_i/C_i with $i \in \{1, \dots, l\}$. Hence

$$q \nmid |F_j \cap N/C_j \cap N|$$

for all $j = 1, \dots, n$. Since $\bigcap_i C_i = M$ and $\bigcap_i (F_i \cap N) = F$, we have that $q \nmid |F/M|$. We have seen above that $Q \subseteq K$ and so $q \nmid |G/K|$. Thus $|G : \mathbf{C}_G(\lambda)|$ is divisible by every prime $p \geq 5$ in $\pi_0(G/K) \cup \pi_0(F/M)$.

Let $Z = N/M$, observe that $W = \text{Irr}(\mathbf{F}(Z)/\Phi(Z))$ is a faithful and completely reducible $Z/\mathbf{F}(Z)$ -module. Since the π_0 -subgroup of KF/F is abelian, there exists $\mu \in W$ such that $|KF/F|_{\pi_0} \mid \mu(1)$.

Now let

$$\beta \in \text{Irr}(G|\mu) \text{ and } \chi \in \text{Irr}(G|\lambda).$$

By the last two paragraphs, $\beta(1)$ is divisible by every prime in $\pi_0(KF/F)$ and $\chi(1)$ is divisible by every prime in $\pi_0(G/K) \cup \pi_0(F/M)$. The conclusion of the lemma is met with $\theta = \beta$ or $\theta = \chi$. \square

The following theorem obtain the known bound of the character version of the Huppert's $\rho - \sigma$ conjecture for G solvable. The bound we obtain here is the same as what Manz and Wolf obtained in [18, Theorems 1.4].

Theorem 6.2. *Let $\rho(G)$ to be those primes that divide the degree of some irreducible character of G , i.e., $p \in \rho(G)$ if and only if p divides $|G : \mathbf{F}(G)|$ or $O_p(G)$ is non-abelian. Let $\sigma(G)$ denote the maximum number of primes dividing the degree of an irreducible character of G . If G is solvable, then $|\rho(G)| \leq 3\sigma(G) + 2$.*

Proof. Let $\mathcal{R} = \{r \text{ prime} \mid O_r(G) \in \text{Syl}_r(G) \text{ and } O_r(G) \text{ is non-abelian}\}$ and $F = \mathbf{F}(G)$. Certainly $\rho(G) \subseteq \pi(G/F) \cup \mathcal{R}$ and by Ito's Theorem [16, 12.33], equality holds.

$\mathbf{F}(G)/\Phi(G)$ is a faithful completely reducible G/F -module. Applying Theorem 6.1 with $G/\Phi(G)$ and $\mathbf{F}(G)/\Phi(G)$ in the role of G and M , there exists $\chi \in \text{Irr}(G)$ with $|\pi_0(\chi(1))| \geq$

$|\pi_0(G/F)|/2$. Hence $\sigma(G) \geq |\pi_0(G/F)|/2$. Now $\prod_{r \in \mathcal{R}} O_r(G) \triangleleft G$ and each $O_r(G)$ is non-abelian. Thus there exists $\eta \in \text{Irr}(G)$ such that $\mathcal{R} \subseteq \pi(\eta(1))$. Since $\sigma(G) \geq \max\{|\mathcal{R}|, |\pi_0(G/F)|/2\}$ and since $\rho(G) = \pi(G/F) \cup \mathcal{R} \subseteq \pi_0(G/F) \cup \mathcal{R} \cup \{2, 3\}$, the result follows. \square

Using the same argument, one may get a similar result for the conjugacy class version of the Huppert $\rho - \sigma$ conjectures (i.e. If G is solvable, then $|\rho^*(G)| \leq 4\sigma^*(G) + 2$).

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